

A SOLUTION TO TINGLEY'S PROBLEM FOR ISOMETRIES BETWEEN THE UNIT SPHERES OF COMPACT C*-ALGEBRAS AND JB*-TRIPLES

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ABSTRACT. Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB*-triples not containing direct summands of rank smaller than or equal to 3. Suppose E has rank greater than or equal to 5. Applying techniques developed in JB*-triple theory, we prove that f admits an extension to a surjective real linear isometry $T : E \rightarrow B$. Among the consequences, we show that every surjective isometry between the unit spheres of two compact C*-algebras A and B (and in particular when $A = K(H)$ and $B = K(H')$) extends to a surjective real linear isometry from A into B . These results provide new examples of infinite dimensional Banach spaces where Tingley's problem admits a positive answer.

1. INTRODUCTION

Let X and Y be normed spaces, whose unit spheres are denoted by $S(X)$ and $S(Y)$, respectively. Suppose $f : S(X) \rightarrow S(Y)$ is a surjective isometry. The so-called *Tingley's problem* asks whether f can be extended to a real-linear (bijective) isometry $T : X \rightarrow Y$ between the corresponding spaces (see [48]). D. Tingley proved in [48, THEOREM, page 377] that every surjective isometry $f : X \rightarrow Y$ between the unit spheres of two finite dimensional spaces satisfies $f(-x) = -f(x)$ for every $x \in S(X)$.

Mankiewicz established in [36] that, given two convex bodies $V \subset X$ and $W \subset Y$, every surjective isometry g from V onto W can be uniquely extended to an affine isometry from X onto Y . Consequently, every surjective isometry between the closed unit balls of two Banach spaces X and Y extends uniquely to a real-linear isometric isomorphism from X into Y .

Many authors have contributed with partial solutions to Tingley's problem for surjective isometries between the unit spheres of concrete Banach spaces. For example, we find affirmative answers to Tingley's problem in the setting of several classical real Banach spaces such as $\ell_p(\Gamma)$ ([12, 14, 15]), L^p -spaces ([39, 40, 41]), and $C(X)$ spaces ([13, 42]). We refer to the monograph [16] for a good survey on the history of Tingley's problem. The solutions we know for this problem are based on interesting geometric ideas (compare [17, 43]). Tingley's problem was solved affirmatively in the case of finite dimensional polyhedral Banach spaces [32], and for the spaces belonging to a special class of Banach spaces called *generalized lush*

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spaces [42]. However, the problem is still open even in the simplest case of $X = Y$ and $\dim X = 2$.

Recently, the study of Tingley's problem on operator algebras was started by the second author of this note in [45], and we now have an affirmative answers in the case of surjective linear isometries between the unit spheres of finite dimensional C^* -algebras (see [46]) and finite von Neumann algebras (cf. [47]). A key ingredient for the methods described in [45, 46, 47] are results describing the facial structure of the unit ball of a C^* -algebra. Using the structure theorem for faces given by C.M. Edwards and G.T. Rüttimann [20], and C.A. Akemann and G.K. Pedersen [1]), we are in position to apply the characterization of the surjective isometries between the unitary groups of two von Neumann algebras given by O. Hatori and L. Molnár [29] to give an affirmative answer to Tingley's problem in the setting of finite von Neumann algebras.

The purpose of this paper is to present some new partial solutions to Tingley's problem in the case of surjective isometries between the unit spheres of two weakly compact JB^* -triples not containing direct summands of rank smaller than or equal to 3. The class of JB^* -triples can be viewed as a Jordan generalization of the category of C^* -algebras. In the wider setting of JB^* -triples, a complete description of the norm closed faces of the closed unit ball was obtained by C.M. Edwards, C.S. Hoskin, F.J. Fernández-Polo and the first author of this note in [18]. Using the result describing the facial structure of the closed unit ball of a JB^* -triple, we present here a completely new method to approach Tingley's problem in the Jordan setting.

The main result in this note (see Theorem 3.13) proves that every surjective isometry, $f : S(E) \rightarrow S(B)$, between the unit spheres of two weakly compact JB^* -triples not containing direct summands of rank smaller or equal than 3, where E has rank greater or equal than 5, extends to a surjective real linear isometry $T : E \rightarrow B$. As an application, we particularly find an affirmative answer to Tingley's problem for surjective isometries between the unit spheres of two compact C^* -algebras (cf. Theorem 3.14). We also prove that every surjective isometry between the unit spheres of two finite dimensional JB^* -triples, where the domain JB^* -triple has rank greater or equal than 5, extends to a surjective real linear isometry between the corresponding spaces (see Theorem 3.19). These results provide new examples of infinite dimensional Banach spaces where Tingley's problem admits a positive answer. The arguments in this note are completely new and independent from those studied in previous references. The results studying the geometry of JB^* -triples, and the Jordan theory itself provide a new point of view to tackle this open problem. Finally, we close the paper with an open problem which cannot be covered by our results.

2. THE CONTRIBUTION OF JORDAN THEORY TO TINGLEY'S PROBLEM

We recall that a proper convex subset of the unit sphere of a normed space X is said to be maximal if C is not contained in any other proper maximal subset of $S(X)$. Let $f : S(X) \rightarrow S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces. Applying Lemma 6.3 in [45] we know that a convex set $C \subset S(X)$ is a maximal convex subset of $S(X)$ if and only if $T(C)$ satisfies the same property as a subset of $S(Y)$.

Henceforth, the closed unit ball of a Banach space X will be denoted by X_1 . We also recall that a convex subset F of X_1 is called a *face* of X_1 if it satisfies the following property: a convex combination $tx + (1-t)y$, with $x, y \in X_1$ and $t \in [0, 1]$, lies in F if and only if x and y belong to F . It is worth to notice that a face F of X_1 contains 0 if and only if $F = X_1$, it is further known that every proper face of X_1 is contained in $S(X)$. Lemma 3.3 in [45] shows how a suitable application of Eidelheit's separation theorem [37, Theorem 2.2.26] proves that for every maximal convex subset C of $S(X)$ there exists φ in $S(X^*)$ satisfying $C = \varphi^{-1}(\{1\}) \cap X_1$, in particular every maximal convex subset C of $S(X)$ is a maximal proper norm closed face of X_1 . Actually, by a result due to the second author of this note we know that a convex subset C of the sphere of a Banach space X is a maximal convex subset of $S(X)$ if and only if it is maximal as a proper face of X_1 (compare [47, Lemma 3.2]).

The arguments in the above two paragraphs can be applied to deduce the following lemma, which has been borrowed from [44, Lemma 3.5].

Lemma 2.1. [44, Lemma 3.5] *Let $f : S(X) \rightarrow S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces, and let C be a convex subset of $S(X)$. Then C is a maximal proper face of X_1 if and only if $f(C)$ is a maximal proper closed face of Y_1 .* \square

The results commented above reveal that, in order to attack Tingley's problem, the facial structure of the corresponding closed unit balls of the spaces plays a fundamental role. The main result in [18] culminate the complete description of the norm closed faces of the closed unit ball of a JB*-triple. Before stating the characterization theorem, we shall recall some definitions needed for our purposes.

2.1. Facial structure of a JB*-triple. Let X be a complex Banach space with dual space X^* . Let X_1 and X_1^* denote the unit balls in X and X^* , respectively. Given subsets F of X_1 and G of X_1^* , we set

$$(1) \quad F' = \{a \in X_1^* : a(x) = 1 \ \forall x \in F\}, \quad G' = \{x \in X_1 : a(x) = 1 \ \forall a \in G\}.$$

Then, F' is a weak*-closed face of X_1^* and G' is a norm closed face of X_1 . The subset F of X_1 is said to be a norm-semi-exposed face of X_1 if F coincides with $(F')'$, and the subset G of X_1^* is said to be a weak*-semi-exposed face of X_1^* if G coincides with $(G')'$.

JB*-triples are Jordan Banach structures which generalize the abstract properties of C*-algebras. The notion of JB*-triple was introduced by W. Kaup in 1983, and provides a precise set of algebraic-analytic axioms to characterize when the open unit ball of a complex Banach space is a bounded symmetric domain (a property of holomorphic nature, see [34] for more details). The concrete definition tells that a JB*-triple is a complex Banach space E equipped with a continuous triple product $\{., ., .\} : E \times E \times E \rightarrow E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables satisfying the following axioms:

- (a) (Jordan Identity) $L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y)$, for all a, b, x, y , in E , where $L(x, y)$ is the linear operator defined by $L(a, b)(z) = \{a, b, z\}$ ($\forall z \in E$);
- (b) The operator $L(a, a)$ is hermitian and has non-negative spectrum;
- (c) $\| \{a, a, a\} \| = \|a\|^3$, for every $a \in E$.

L.A. Harris established in [28] that the open unit ball of a C^* -algebra A is a bounded symmetric domain, and hence every C^* -algebra A lies in the category of JB^* -triples. It is known that the triple product

$$(2) \quad (a, b, c) \mapsto \{a, b, c\} = 1/2(ab^*c + cb^*a), \quad (a, b, c \in A),$$

satisfies all the axioms in the definition of JB^* -triple. The category of JB^* -triples contains many other examples of Banach spaces which are not C^* -algebras. For example, given two complex Hilbert spaces H_1 and H_2 , the triple product defined in (2) equips the space $L(H_1, H_2)$, of all bounded linear operators between H_1 and H_2 , with a structure of JB^* -triple. In particular, every complex Hilbert space is a JB^* -triple. JB^* -triples of the form $L(H_1, H_2)$ are called Cartan factors of type 1. Clearly the space $K(H_1, H_2)$, of all compact operators from H_1 into H_2 is a JB^* -subtriple of $L(H_1, H_2)$. Additional examples can be given by the rest of Cartan factors, which are defined as follows. Let j be a conjugation (i.e. a conjugate linear isometry of period 2) on a complex Hilbert space H . The assignment $x \mapsto x^t := jx^*j$ defines a linear involution on $L(H)$. A Cartan factor of type 2 (respectively, of type 3) is a complex Banach space which coincides with the JB^* -subtriple of $L(H)$ of all t -skew-symmetric (respectively, t -symmetric) operators.

A Cartan factor of type 4, is a complex Hilbert space provided with a conjugation $x \mapsto \bar{x}$, where triple product and the norm are given by

$$(3) \quad \{x, y, z\} = (x|y)z + (z|y)x - (x|\bar{z})\bar{y},$$

and $\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |(x|\bar{x})|^2}$, respectively. All we need to know about Cartan factors of types 5 and 6 is that they are finite dimensional (see [7, Example 2.5.31] for additional details).

One of the most interesting properties of JB^* -triples was established by W. Kaup in [34] and assures that a (complex) linear bijection between JB^* -triples is a triple isomorphism if and only if it is an isometry (compare [34, Proposition 5.5] or [23, Theorem 2.2]). The same conclusion is no longer true for real linear isometries between JB^* -triples, for such a mapping T we only know that T preserves cubes, that is, $T(\{x, x, x\}) = \{T(x), T(x), T(x)\}$ (compare [9, 8, 31] and [22]). The conclusion in Tingley's theorem makes more useful the study of real linear isometric surjections between JB^* -triples.

A JB^* -triple which is also a dual Banach space is called a JBW^* -triple. For example, every von Neumann algebra is a JBW^* -triple. All Cartan factors are JBW^* -triples. Every JBW^* -triple admits a unique isometric predual, and its triple product is separately weak* continuous [3]. Furthermore, the second dual of a JB^* -triple E is a JBW^* -triple under a triple product extending the product of E [10].

An element e in a JBW^* -triple E is called *tripotent* if $\{e, e, e\} = e$. The set of all tripotents in E will be denoted by $\mathcal{U}(E)$. In this case the eigenvalues of the operator $L(e, e)$ are precisely 0, 1/2 and 1 and E decomposes as the direct sum of the corresponding eigenspaces, that is,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e, e)$ (compare [7, Definition 1.2.37]). This decomposition is called the *Peirce decomposition* of E with respect

to the tripotent e , and the projection of E onto $E_i(e)$, which is denoted by $P_i(e)$, is called the Peirce i projection.

The so-called *Peirce arithmetic* asserts that, for $k, j, l \in \{0, 1, 2\}$ we have

$$\{E_k(e), E_j(e), E_l(e)\} \subseteq E_{k-j+l}(e),$$

if $k - j + l \in \{0, 1, 2\}$, and is equal to $\{0\}$ otherwise. Moreover

$$\{E_0(e), E_2(e), E\} = \{0\} = \{E_2(e), E_0(e), E\}.$$

A tripotent e in E is called *complete*, or *unitary*, or *minimal* if $E_0(E) = 0$, or $E_2(e) = E$, or $E_2(e) = \mathbb{C}e \neq \{0\}$, respectively. We say that e has *finite rank* if it can be written a sum of finitely many mutually orthogonal minimal tripotents.

Every hermitian element in a C*-algebra defines a commutative C*-algebra and that allows us to define a local functional calculus. However, for non-normal elements the spectral resolutions and the functional calculus is hopeless. There is a certain advantage in the setting of JB*-triples. Accordingly to the standard notation, given an element x in a JB*-triple E , we shall write $x^{[1]} := x$, $x^{[3]} := \{x, x, x\}$, and $x^{[2n+1]} := \{x, x, x^{[2n-1]}\}$, ($n \in \mathbb{N}$), while the symbol E_x will stand for the JB*-subtriple generated by the element x , that is, the norm closure of the linear span of all odd powers $x^{[2n-1]}$ ($n \in \mathbb{N}$) of x . It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$, such that $\Omega \cup \{0\}$ is compact and $\|x\| \in \Omega$, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that there exists a triple isomorphism Ψ from E_x onto $C_0(\Omega)$, satisfying $\Psi(x)(t) = t$ ($t \in \Omega$) (cf. [34, Corollary 1.15]).

Elements a, b in a JB*-triple E are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. It is known that $a \perp b \Leftrightarrow \{a, a, b\} = 0 \Leftrightarrow \{b, b, a\} = 0 \Leftrightarrow b \perp a$; $\Leftrightarrow E_a \perp E_b$ (see, for example, [6, Lemma 1]). Let e be a tripotent in E . It follows from the Peirce arithmetic that $a \perp b$ for every $a \in E_2(e)$ and every $b \in E_0(e)$. It is known that $a \perp b$ in E implies that $\|\lambda a + \mu b\| = \max\{\|\lambda a\|, \|\mu b\|\}$ (compare [27, Lemma 1.3(a)]).

The relation “being orthogonal” is applied to define a partial order on the set of tripotents of a JB*-triple E defined by the following: given $u, e \in \mathcal{U}(E)$, we write $u \leq e$ if $e - u$ is a tripotent in E and $e - u \perp u$. The following fact will be needed later. Let e and u be tripotents in a JB*-triple E , then

$$(4) \quad u \perp e \Leftrightarrow u \pm e \text{ are a tripotents}$$

(see [31, Lemma 3.6]).

When a is a norm one element in a JBW*-triple W , the sequence $(a^{[2n-1]})$ converges in the weak* topology of W to a tripotent (called the *support tripotent* of a) $s(a)$ in W (compare [20, Lemma 3.3] or [18, page 130]). It is known that $a = s(a) + P_0(s(a))(a)$. For a norm one element a in a JB*-triple E , $s(a)$ will denote the support tripotent of a in E^{**} .

Following [21], a tripotent e in the second dual of a JB*-triple E is said to be *compact- G_δ* if there exists a norm one element a in E satisfying $s(a) = e$. A tripotent e in E^{**} is called *compact* if $e = 0$ or it is the infimum of a decreasing net of compact- G_δ tripotents in E^{**} . The set of all compact tripotents in E^{**} will be denoted by $\mathcal{U}_c(E^{**})$.

According to [25], a tripotent $e \in E^{**}$ is said to be *bounded* if there exists a norm one element a in E such that $P_2(e)(a) = e$. It is known that, in these circumstances, $P_1(e)(a) = 0$. We shall write $e \leq_T a$ when a satisfies the above conditions. The relation \leq_T is consistent with the natural partial order on the set of tripotents, that is, for any two tripotents e and u we have $e \leq u$ if and only if $e \leq_T u$. Following the same reference, a tripotent e in E^{**} satisfying that $E_0^{**}(e) \cap E$ is weak* dense in $E_0^{**}(e)$ is called *closed* relative to E .

The following characterization of compact tripotents in the second dual of a JB*-triple has been borrowed from [25, Theorem 2.6] (see also [26, Theorem 3.2]), and will be applied later.

Theorem 2.2. [25, Theorem 2.6] *Let E be a JB*-triple. Then a tripotent e in E^{**} is compact if and only if e is closed and bounded.* \square

After having introduced the necessary concepts, the norm closed faces of the closed unit ball of a JB*-triple E can be characterized in terms of the compact tripotents in E^{**} via the next theorem, which due to Edwards, Fernández-Polo, Hoskin and the first author of this note (see [18]).

Theorem 2.3. *Let E be a JB*-triple, and let F be a non-empty norm closed face of the unit ball E_1 in E . Then, there exists uniquely a compact tripotent u in E^{**} such that*

$$F = (u + E_0^{**}(u)_1) \cap E = (\{u\}_r)_r,$$

where $E_0^{**}(u)_1$ is the unit ball in the Peirce-zero space $E_0^{**}(u)$ in E^{**} and $(\{u\}_r)_r$ is the norm-semi-exposed face of E_1 corresponding to u (as defined in (1)). Furthermore, the mapping $u \mapsto (\{u\}_r)_r$ is an anti-order isomorphism from $\mathcal{U}_c(E^{**})$ onto the complete lattice $\mathcal{F}_n(E_1)$ of norm closed faces of E_1 . \square

Let E be a JB*-triple. It is not obvious but every minimal tripotent, actually every finite rank tripotent in E^{**} is compact (see [5, Theorem 3.4]). It is known that the minimal elements of the set $\mathcal{U}_c(E^{**})$ are precisely the minimal tripotents of E^{**} (compare the comments before [5, Corollary 3.5]). We shall see in the next subsection that finite rank tripotents can be also applied to characterize compact JB*-triples.

2.2. Weakly compact JB*-triples. The facial structure of the closed unit ball is much simpler in the case of a compact JB*-triple. For this reason we briefly survey the basic notions of compact JB*-triples. Motivated by the studies on compact C*-algebras published by Alexander and Ylinen (see [2, 49]), Bunce and Chu introduced and studied in [4] the notions of compact and weakly compact JB*-triples.

An element x in a JB*-triple E is called *compact* (respectively, *weakly compact*) if the operator $Q(x) : E \rightarrow E$, $z \mapsto \{x, z, x\}$ is compact (respectively, weakly compact). The JB*-triple E is *compact* (respectively, *weakly compact*) if every element in E is compact (respectively, weakly compact).

The connections between compact JB*-triples and finite rank tripotents are very useful to obtain a concrete representation of every compact JB*-triple. Following the notation in [4], the Banach subspace of a JB*-triple E generated by all its minimal tripotents will be denoted by $K(E)$. It is known that $K(E)$ is a (norm closed) triple ideal of E which coincides with the set of all weakly compact elements in E (see [4, Proposition 4.7]). In order to have a more concrete representation, we recall that for each Cartan factor C the elementary JB*-triple associated with C

is precisely $K(C)$. That is, there are six different types of elementary JB*-triples, denoted by K_i ($i = 1, \dots, 6$), which defined as follows: $K_1 = K(H, H')$ (the compact operators between two complex Hilbert spaces H and H'); $K_i = C_i \cap K(H)$ for $i = 2, 3$, and $K_i = C_i$ for $i = 4, 5, 6$. The following structure theorem was established by Bunce and Chu in [4].

Theorem 2.4. [4, Lemma 3.3 and Theorem 3.4] *Let E be a JB* triple. Then E is weakly compact if and only if one of the following statement holds:*

- a) $K(E^{**}) = K(E)$;
- b) $K(E) = E$;
- c) E is a c_0 -sum of elementary JB*-triples. □

Let E be a reflexive JB*-triple. Clearly, every compact tripotent in E^{**} lies in E . The same conclusion holds when E is a weakly compact JB*-triple. More concretely, suppose E is a weakly compact JB*-triple and e is a compact tripotent in E^{**} . By Theorem 2.2 there exists a norm one element a in E such that $e \leq_T a$. An application of Theorem 2.4 assures that e must be a finite rank tripotent in E .

Corollary 2.5. *Let E be a weakly compact JB*-triple. Then every compact tripotent in E^{**} is a finite rank tripotent in E .* □

3. SOLUTION TO TINGLEY'S PROBLEM FOR WEAKLY COMPACT JB*-TRIPLES

This section is devoted to obtain a complete solution to Tingley's problem in the case of surjective isometries between the unit spheres of arbitrary weakly compact JB*-triples and several consequences in the setting of C*-algebras and operator spaces. We begin with a key proposition which improves the conclusion of Lemma 2.1.

Henceforth, given a vector x_0 in a Banach space X , the translation with respect to x_0 will be denoted by \mathcal{T}_{x_0} .

Proposition 3.1. *Let E and B be weakly compact JB*-triples, and suppose that $f : S(E) \rightarrow S(B)$ is a surjective isometry. Then the following statements hold:*

- (a) *For each minimal tripotent e_1 in E there exists a unique minimal tripotent u_1 in B such that $f((e_1 + E_0^{**}(e_1)_1) \cap E) = (u_1 + B_0^{**}(u_1)_1) \cap B$;*
- (b) *The restriction of f to each maximal proper face of E_1 is an affine function;*
- (c) *For each minimal tripotent e_1 in E there exists a unique minimal tripotent u_1 in B such that $f(e_1) = u_1$;*
- (d) *f maps norm closed proper faces of E_1 to norm closed faces of B_1 .*

Proof. Let $M \subseteq S(E)$ be a maximal proper (norm closed) face of E_1 . Lemma 2.1 implies that $f(M)$ is maximal proper (norm closed) face of E_1 . Combining Theorem 2.3 and subsequent comments with Corollary 2.5, we deduce the existence of minimal tripotents $e_1 \in E$ and $u_1 \in B$ such that $M = (e_1 + E_0^{**}(e_1)_1) \cap E$ and $f(M) = (u_1 + B_0^{**}(u_1)_1) \cap B$. Since by Theorem 2.3 and Corollary 2.5 minimal tripotents in E and B are in one-to-one correspondence with the maximal proper (norm closed) face of E_1 and B_1 , respectively, the statement in (a) follows from the above arguments.

We also know that, under the above assumptions, $E_0^{**}(e_1) \cap E = E_0(e_1)$ and $B_0^{**}(u_1) \cap B = B_0(u_1)$ are weak*-dense (norm-closed) subspaces of $E_0^{**}(w_1)$ and $B_0^{**}(u_1)_1$, respectively, whose closed unit balls are precisely $E_0(e_1)_1$ and $B_0(u_1)_1$,

respectively. The mapping $f_{e_1} = \mathcal{T}_{u_1}^{-1}|_{f(M)} \circ f|_M \circ \mathcal{T}_{e_1}|_{E_0(e_1)_1}$ is a surjective isometry from $E_0(e_1)_1$ onto $B_0(u_1)_1$. Mankiewicz's theorem (see [36]) assures the existence of a surjective real linear isometry $T_{e_1} : E_0(e_1) \rightarrow B_0(u_1)$ such that $f_{e_1} = T_{e_1}|_{S(E_0(e_1))}$. Since translations and linear isometries are affine functions, the identity $f|_M = \mathcal{T}_{e_1}^{-1}|_M \circ T_{e_1}|_{S(E_0(e_1))} \circ \mathcal{T}_{u_1}|_{B_0(u_1)_1 B}$ proves that $f|_M$ is an affine function, which proves (b). We also know that $f(e_1) = u_1$, which gives (c).

We shall finally prove (d). Let F be a norm closed proper face of E_1 . As before, an appropriate combination of Theorem 2.3 and Corollary 2.5 implies the existence of a finite rank tripotent e in E such that $F = (e + E_0^{**}(e)_1) \cap E = \{e\}_\eta$. Take a minimal tripotent e_1 such that $e_1 \leq e$. Since $F \subseteq \{e_1\}_\eta$, the latter is a maximal proper face of E_1 , and, by (b), $f|_{\{e_1\}_\eta}$ is affine, we deduce that $f(F)$ is a convex subset of $S(B)$. Applying that f is an isometry we can easily see that $f(F)$ is closed. Suppose that $ta + (1-t)b = f(c)$, where $a, b \in S(B)$, $c \in F$ and $t \in (0, 1)$. Pick a maximal proper face $M_1 \subset S(B)$ such that $f(\{e\}_\eta) = M_1$. Since M_1 is a norm closed face of B_1 , it follows that $a, b \in M_1$. Applying statement (b) to $f^{-1}|_{M_1}$ we get $tf^{-1}(a) + (1-t)f^{-1}(b) = f^{-1}(ta + (1-t)b) = c \in F$, and thus $a, b \in f(F)$, because F is a face. This shows that $f(F)$ is a norm closed proper face of B_1 . \square

The next result, whose proof is essentially based on the arguments given in the proof previous proposition, goes deeper in the conclusions given above.

Proposition 3.2. *Let E and B be weakly compact JB^* -triples, and suppose that $f : S(E) \rightarrow S(B)$ is a surjective isometry. Then the following statements hold:*

- (a) *For each finite rank tripotent e in E there exists a unique finite rank tripotent u in B such that $f((e + E_0^{**}(e)_1) \cap E) = (u + B_0^{**}(u)_1) \cap B$;*
- (b) *The restriction of f to each norm closed face of E_1 is an affine function;*
- (c) *For each finite rank tripotent e in E there exists a unique finite rank tripotent u in B and a surjective real linear isometry $T_e : E_0(e) \rightarrow B_0(u)$ such that*

$$f(e + x) = u + T_e(x),$$

for every $x \in E_0(e)_1$;

- (d) *For each finite rank tripotent e in E there exists a unique finite rank tripotent u in B such that $f(e) = u$.*

Proof. (a) Follows from Theorem 2.3, Corollary 2.5 and Proposition 3.1(d).

Let e be a finite rank tripotent in E . By (a) for each finite rank tripotent e in E there exists a unique finite rank tripotent u in B such that

$$f((e + E_0^{**}(e)_1) \cap E) = (u + B_0^{**}(u)_1) \cap B.$$

Arguing as in the proof of Proposition 3.1, the mapping $f_e = \mathcal{T}_u^{-1}|_{f(\{e\}_\eta)} \circ f|_{\{e\}_\eta} \circ \mathcal{T}_e|_{E_0(e)_1}$ is a surjective isometry from $E_0(e)_1$ onto $B_0(u)_1$. By Mankiewicz's theorem (see [36]) there exists a surjective real linear isometry $T_e : E_0(e) \rightarrow B_0(u)$ such that $f_e = T_e|_{S(E_0(e))}$. Therefore $f|_{\{e\}_\eta} = \mathcal{T}_u^{-1}|_{\{e\}_\eta} \circ T_e|_{S(E_0(e))} \circ \mathcal{T}_e|_{B_0(u)_1}$ is an affine function, which proves (b). We further know that

$$f(e + x) = u + T_e(x),$$

for every $x \in E_0(e)_1$ and $f(e) = u$, which gives (c) and (d). \square

We recall that the rank of a finite rank tripotent e in a JB^* -triple E is the unique natural k satisfying that e writes as a sum of k mutually orthogonal minimal tripotents in E . A subset S of a JB^* -triple E is called *orthogonal* if $0 \notin S$ and

$x \perp y$ for every $x \neq y$ in S . The minimal cardinal number r satisfying $\text{card}(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the *rank* of E .

We are now in position to prove that a surjective isometry between the unit spheres of two weakly compact JB*-triples maps tripotents of rank k to tripotents of rank k . We begin with some technical lemmas. The first one is a geometric version of (4) (see [31, Lemma 3.6]), a version of which was considered in [24].

We recall that, given a subset S of the closed unit ball of a Banach space X , the set of all contractive perturbations of S , $\text{cp}(S)$, is defined by

$$\text{cp}(S) = \{x \in X : \|x \pm s\| \leq 1, \text{ for every } s \in S\} \quad (\text{see [24]}).$$

Lemma 3.3. [24] *Let e be a tripotent in a JB*-triple E . Let x be an element in E_1 satisfying $\|e \pm x\| = 1$. Then $x \perp e$.*

Proof. The identity (6) in [24] (see also [19, Corollary 4.3]) shows that

$$\{e\}^\perp \cap E_1 = \text{cp}(\{e\}),$$

where $\{e\}^\perp = \{x \in E : e \perp x\}$. An element x in the hypothesis of the Lemma belongs to $\{e\}^\perp \cap E_1$ and hence $x \perp e$. \square

Lemma 3.4. *Let e and w be finite rank tripotents in a weakly compact JB*-triple E not containing direct summands of rank smaller or equal than 3. Suppose that e is minimal and $\|e - w\| = 2$. Then $w = -e + P_0(e)(w)$.*

Proof. By the structure theory of weakly compact JB*-triples (see Theorem 2.4), there is no loss of generality in assuming that E is an elementary JB*-triple of rank bigger or equal than 4.

We assume first that $E = K(H_1, H_2)$, where H_1 and H_2 are complex Hilbert spaces (i.e. E is an elementary JB*-triple of type K_1). It is well known that in this case, $e = \xi \otimes \eta$ and $w = \sum_{j=1}^m \zeta_j \otimes \vartheta_j$, where $\{\zeta_1, \dots, \zeta_m\}$ and $\{\vartheta_1, \dots, \vartheta_m\}$ are orthonormal systems in H_1 and H_2 , respectively. The element $w - e$ is a finite rank operator with $\|e - w\| = 2$. Then there exists a norm one element $h \in H_1$ such that $\|(e - w)(h)\| = 2$. If $|\langle h, \eta \rangle| < 1$ then

$$2 = \|(e - w)(h)\| \leq \|w(h)\| + \|\langle h, \eta \rangle \xi\| < 2,$$

which is impossible. So, $|\langle h, \eta \rangle| = 1$, and hence $h = \langle h, \eta \rangle \eta$. Similarly, if $|\langle h, \vartheta_j \rangle| < 1$, for every $1 \leq j \leq m$, we would have $2 = \|(e - w)(h)\| \leq \|e(h)\| + \|w(h)\| < 2$, which is impossible. Therefore there exists $j_1 \in \{1, \dots, m\}$ such that $|\langle h, \vartheta_{j_1} \rangle| = 1$ and $h = \langle h, \vartheta_{j_1} \rangle \vartheta_{j_1}$. In particular $\eta = \lambda \vartheta_{j_1}$ for a suitable λ in \mathbb{C} with $|\lambda| = 1$.

Since $\|e^* - w^*\| = 2$ it follows from the above arguments the existence of $j_2 \in \{1, \dots, m\}$ such that $\xi = \mu \zeta_{j_2}$ for a suitable μ in \mathbb{C} with $|\mu| = 1$. Therefore $e = \bar{\lambda} \mu \zeta_{j_2} \otimes \vartheta_{j_1}$, and the condition

$$2 = \|(e - w)(h)\| = \|\bar{\lambda} \mu \langle h, \vartheta_{j_1} \rangle \zeta_{j_2} - \langle h, \vartheta_{j_1} \rangle \zeta_{j_1}\|$$

implies that $j_1 = j_2$, $\bar{\lambda} \mu = 1$, and hence $w = -e + P_0(e)(w)$. The cases in which E is of type K_2 or K_3 follow by similar arguments. \square

Since every compact C*-algebra can be written as a c_0 sum of C*-algebras of the form $K(H_i)$, where each H_i is a complex Hilbert space, the proof of the previous lemma actually shows the following statement.

Lemma 3.5. *Let e and w be finite rank tripotents in a compact C^* -algebra A . Suppose that e is minimal and $\|e - w\| = 2$. Then $w = -e + P_0(e)(w)$. \square*

The following result is a generalization of Tingley's theorem in the setting of weakly compact JB^* -triples.

Theorem 3.6. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB^* -triples. We assume that E and B do not contain direct summands of rank smaller or equal than 3. Suppose e is a finite rank tripotent in E . Then $f(-e) = -f(e)$. Furthermore, if e_1, \dots, e_m are mutually orthogonal minimal tripotents in E , then*

$$f(e_1 + \dots + e_m) = f(e_1) + \dots + f(e_m).$$

Proof. We shall first prove that

$$(5) \quad f(-e_1) = -f(e_1), \text{ for every minimal tripotent } e_1 \in E.$$

Indeed, by Proposition 3.1 $v_1 = f(e_1)$ and $w_1 = f(-e_1)$ are minimal tripotents in B . By the assumptions on f we have

$$\|v_1 - w_1\| = \|f(e_1) - f(-e_1)\| = \|e_1 + e_1\| = 2,$$

which, via Lemma 3.4, proves that $w_1 = -v_1 + P_0(v_1)(w_1)$. However, w_1 being minimal implies that $w_1 = -v_1$, which proves (5).

Let e_1, \dots, e_m be mutually orthogonal minimal tripotents in E . Proposition 3.1 assures that $v_j = f(e_j)$ is a minimal tripotent for every $1 \leq j \leq m$. We also know from (5) that $f(-e_j) = -f(e_j)$, for every such j .

We claim that

$$(6) \quad v_j \perp v_k, \text{ for every } j \neq k.$$

To see this, for each $j \neq k$, we observe that

$$\|v_j \pm v_k\| = \|f(e_j) \pm f(e_k)\| = \|f(e_j) - f(\pm e_k)\| = \|e_j \pm e_k\| = 1,$$

and hence Lemma 3.3 implies that $v_j \perp v_k$.

Let $w = f(-e_1 - \dots - e_m)$ and $v = f(e_1 + \dots + e_m)$. We shall show that $w = -v$. Proposition 3.2 implies that w and v are finite rank tripotents. Fix $1 \leq j \leq m$. It follows from the hypothesis that

$$\|w - v_j\| = \|-e_1 - \dots - e_m - e_j\| = 2.$$

An application of Lemma 3.4 shows that $w = -v_j + P_0(v_j)(w)$, for every $1 \leq j \leq m$. Having in mind that, by (6), v_1, \dots, v_m are mutually orthogonal, we can easily deduce that

$$\begin{aligned} f(-e_1 - \dots - e_m) &= w = -v_1 - \dots - v_m + P_0(v_1 + \dots + v_m)(w) \\ &= -v + P_0(v_1 + \dots + v_m)(w) = -f(e_1 + \dots + e_m) + P_0(v_1 + \dots + v_m)(w). \end{aligned}$$

This shows that $f(-e_1 - \dots - e_m) = w$ has rank greater or equal than m . If w had rank strictly bigger than m , it would follow from the above arguments applied to f^{-1} that $f^{-1}(w) = -e_1 - \dots - e_m$ had rank strictly bigger than m , which is impossible. Therefore $P_0(v_1 + \dots + v_m)(w) = 0$ and hence

$$f(-e_1 - \dots - e_m) = -f(e_1) - \dots - f(e_m).$$

The above arguments also show that $f(e_1 + \dots + e_m) = f(e_1) + \dots + f(e_m)$, and thus $f(-e_1 - \dots - e_m) = -f(e_1 + \dots + e_m)$. \square

When in the proof of Theorem 3.6, Lemma 3.4 is replaced with Lemma 3.5 we obtain the following result.

Theorem 3.7. *Let $f : S(A) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two compact C^* -algebras. Suppose e is a finite rank tripotent in E . Then $f(-e) = -f(e)$. Furthermore, if e_1, \dots, e_m are mutually orthogonal minimal tripotents in E , then*

$$f(e_1 + \dots + e_m) = f(e_1) + \dots + f(e_m).$$

□

The next result is a direct consequence of Theorems 3.6 and 3.7.

Corollary 3.8. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB^* -triples. We assume that E and B do not contain direct summands of rank smaller or equal than 3. Then the following statements hold:*

- (a) *If e_1 and e_2 are two orthogonal finite rank tripotents in E . Then $f(e_1)$ and $f(e_2)$ are two orthogonal finite rank tripotents in B with $f(e_1 + e_2) = f(e_1) + f(e_2)$.*
- (b) *If e is a rank k tripotent in E then $f(e)$ is a rank k tripotent in B .*
- (c) *The rank of E and the rank of B coincide.*

Furthermore, the same statements hold when E and B are compact C^ -algebras.*

Our next proposition plays a fundamental role in our results.

Proposition 3.9. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB^* -triples. Let A be a JB^* -subtriple of E , and suppose that $T : A \rightarrow B$ is a bounded linear operator such that $T(u) = f(u)$ for every finite rank tripotent $u \in A$. Then $f(x) = T(x)$, for every $x \in S(A)$.*

Proof. Let x be an element in $S(A)$. Since every JB^* -subtriple of a weakly compact JB^* -triple is weakly compact (see [4, Lemma 3.2]), it follows that A is weakly compact. Therefore, we can assure that x can be approximated in norm by an element of the form $z = \sum_{j=1}^m \lambda_j e_j$, where e_1, \dots, e_m are mutually orthogonal minimal tripotents in A and $0 < \lambda_j \leq \|x\| = 1$ for every j (see [4, Remark 4.6]). The element z can be chosen with the condition that $\|z\| = 1$ (that is, $\lambda_j = 1$ for some $j \in \{1, \dots, m\}$). By the norm-density of this type of elements and the continuity of f and T , the desired conclusion follows as soon as we prove that $f(z) = T(z)$, for every z as above.

Let us write such an element z in the form

$$z = u_1 + \dots + u_k + \sum_{j=k+1}^m \lambda_j u_j = u + \sum_{j=k+1}^m \lambda_j u_j,$$

where u_1, \dots, u_m are mutually orthogonal minimal tripotents in A (and in E), $u = u_1 + \dots + u_k$, and $0 < \lambda_j < 1$. Let \mathcal{C} denote the JB^* -subtriple of A generated by $\{u_{k+1}, \dots, u_m\}$. Clearly, \mathcal{C} is JB^* -triple isomorphic to a unital C^* -algebra of

dimension $m - k$ and $\sum_{j=k+1}^m \lambda_j u_j \in \mathcal{C}$ with $\left\| \sum_{j=k+1}^m \lambda_j u_j \right\| < 1$. By the Russo-Dye theorem (see [38]), or by the strengthened version proved by Kadison and Pedersen

in [33], the element $\sum_{j=k+1}^m \lambda_j u_j$ is the mean of a finite number of unitary elements in \mathcal{C} , that is

$$\sum_{j=k+1}^m \lambda_j u_j = \frac{1}{m_1}(w_1 + \dots + w_{m_1}),$$

where w_1, \dots, w_{m_1} are unitary elements in the C^* -algebra \mathcal{C} . It is not hard to see that, defining $\tilde{w}_j = u + w_j$, $j \in \{1, \dots, m_1\}$, it follows that $\tilde{w}_1, \dots, \tilde{w}_{m_1}$ are tripotents in A (and in E) and

$$(7) \quad z = \frac{1}{m_1}(\tilde{w}_1 + \dots + \tilde{w}_{m_1}).$$

Clearly $z, \tilde{w}_1, \dots, \tilde{w}_{m_1}$ lie in a proper norm closed face of E_1 , so applying Proposition 3.2(b) and the hypothesis we get

$$f(z) = \frac{1}{m_1}(f(\tilde{w}_1) + \dots + f(\tilde{w}_{m_1})) = \frac{1}{m_1}(T(\tilde{w}_1) + \dots + T(\tilde{w}_{m_1})) = T(z),$$

which concludes the proof. \square

Corollary 3.10. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB^* -triples not containing direct summands of rank smaller or equal than 3. Let e be a finite rank tripotent in E , $u = f(e)$, and let $T_e : E_0(e) \rightarrow B_0(u)$ be the surjective real linear isometry given by Proposition 3.2(c). Then $f(x) = T_e(x)$ for all $x \in S(E_0(e))$. The conclusion also holds when E and B are compact C^* -algebras.*

Proof. Let e_1 be a rank-one tripotent in $E_0(e)$. By Corollary 3.8 we have $f(e \pm e_1) = f(e) \pm f(e_1)$, and from Proposition 3.2(c) we also know that

$$f(e) \pm f(e_1) = f(e \pm e_1) = f(e) + T_e(\pm e_1),$$

which implies that $T_e(e_1) = f(e_1)$. A new application of Corollary 3.8 proves that

$$(8) \quad T_e(u) = f(u),$$

for every finite rank tripotent u in $E_0(e)$. Since $E_0(e)$ is a JB^* -subtriple of E , the desired conclusion follows from (8) and Proposition 3.9. \square

We shall need some additional properties of the real linear isometries T_e given by Proposition 3.2(c).

Lemma 3.11. *$f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB^* -triples not containing direct summands of rank smaller or equal than 3, or between two compact C^* -algebras. Let e_1 and e_2 be two orthogonal finite rank tripotents in E , and let T_{e_1} and T_{e_2} be the maps given by Proposition 3.2(c). Then*

$$T_{e_1}(x) = T_{e_2}(x) \text{ for all } x \in E_0(e_1) \cap E_0(e_2).$$

Proof. Let x be an element in $E_0(e_1) \cap E_0(e_2)$ with $\|x\| \leq 1$. Let $u_j = f(e_j)$. We deduce, via Proposition 3.2(c), that

$$u_2 + T_{e_2}(e_1 + x) = f(e_1 + e_2 + x) = u_1 + T_{e_1}(e_2 + x),$$

and by Corollary 3.8

$$u_1 + u_2 = f(e_1) + f(e_2) = f(e_1 + e_2) = u_1 + T_{e_1}(e_2) = u_2 + T_{e_2}(e_1).$$

Therefore, $u_1 = T_{e_2}(e_1)$, $T_{e_1}(e_2) = u_2$, $T_{e_1}(x) = T_{e_2}(x)$, which proves the statement. \square

We can state now a first answer to Tingley's problem in the case of weakly compact JB^* -triples which are expressed as a sum of at least two elementary JB^* -triples.

Theorem 3.12. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB^* -triples not containing direct summands of rank smaller or equal than 3 (respectively, between two compact C^* -algebras). Suppose that $E = \bigoplus_{j \in J} K_j$, where $\#J \geq 2$, and every K_j is an elementary JB^* -triple (respectively, each K_j coincides with $K(H_j)$ for a suitable complex Hilbert space H_j). Then there exists a surjective real linear isometry $T : E \rightarrow B$ satisfying $T|_{S(E)} = f$.*

Proof. Since $\#J \geq 2$, we can pick two different subindexes j_1 and j_2 in J . Let $e_{j_1} \in K_{j_1}$ and $e_{j_2} \in K_{j_2}$ be finite rank tripotents, and let $T_{e_j} : E_0(e_j) \rightarrow B_0(u_j)$ be the surjective real linear isometry given by Proposition 3.2(c), where $u_j = f(e_j)$. By Corollary 3.10 we know that $f(x) = T_{e_j}(x)$ for all $x \in S(E_0(e_j))$.

It is easy to see that

$$(9) \quad E = K_{j_1} \oplus K_{j_2} \oplus (\bigoplus_{j \neq j_1, j_2}^{c_0} K_j),$$

with $\bigoplus_{j \neq j_1, j_2}^{c_0} K_j \subseteq E_0(e_{j_1}) \cap E_0(e_{j_2})$.

Lemma 3.11 proves that

$$(10) \quad T_{e_{j_1}}(x) = T_{e_{j_2}}(x) \text{ for all } x \in E_0(e_{j_1}) \cap E_0^{**}(e_{j_2}).$$

Let us define a mapping $T : E \rightarrow B$ given by

$$T(x) = T_{e_{j_2}}(x_{j_1}) + T_{e_{j_1}}(x_{j_2}) + T_{e_{j_1}}(x_0),$$

where $x = x_{j_1} + x_{j_2} + x_0$, $x_{j_1} \in K_{j_1}$, $x_{j_2} \in K_{j_2}$, $x_0 \in \bigoplus_{j \neq j_1, j_2}^{c_0} K_j$, with respect to (9). The mapping T is well defined thanks to (10) and the uniqueness of the decomposition in (9). The same argument and the real linearity of $T_{e_{j_1}}$ and $T_{e_{j_2}}$ prove that T is real linear. Clearly T is bounded with $\|T\| \leq 3$.

Every finite rank tripotent in E is of the form $u = u_{j_1} + u_{j_2} + u_0$, where u_{j_1} , u_{j_2} and u_0 are finite rank tripotents in K_{j_1} , K_{j_2} , and $\bigoplus_{j \notin \{j_1, j_2\}}^{c_0} K_j \subseteq E_0(e_{j_1}) \cap E_0(e_{j_2})$, respectively. Now, by Corollary 3.8 we have

$$\begin{aligned} f(u) &= f(u_{j_1} + u_{j_2} + u_0) = f(u_{j_1} + u_{j_2}) + f(u_0) = f(u_{j_1}) + f(u_{j_2}) + f(u_0) \\ &= (\text{by Corollary 3.10}) = T_{e_{j_2}}(u_{j_1}) + T_{e_{j_1}}(u_{j_2}) + T_{e_{j_1}}(u_0) = T(u). \end{aligned}$$

We have therefore shown that $f(u) = T(u)$ for every finite rank tripotent u in E . Finally Proposition 3.9 assures that $T(x) = f(x)$ for every $x \in S(E)$, and in particular T is a real linear surjective isometry. \square

Many interesting consequences can be derived from the previous theorem. For example, suppose that H_1, H_2, H_3 and H_4, H'_1, H'_2, H'_3 , and H'_4 are complex Hilbert spaces and $f : S(K(H_1, H_2) \oplus^\infty K(H_3, H_4)) \rightarrow S(K(H'_1, H'_2) \oplus^\infty K(H'_3, H'_4))$ is a surjective real linear isometry. Then there exists a real linear surjective isometry $T : K(H_1, H_2) \oplus^\infty K(H_3, H_4) \rightarrow K(H'_1, H'_2) \oplus^\infty K(H'_3, H'_4)$ such that

$$T|_{S(K(H_1, H_2) \oplus^\infty K(H_3, H_4))} = f.$$

After Theorem 3.12 above and the structure theory of weakly compact JB^* -triples (see Theorem 2.4), the study of surjective isometries between the unit spheres

of two compact JB^* -triples can be reduced to the study of surjective isometries between the unit spheres of two elementary JB^* -triples.

We are now in position to prove the second main result of this note.

Theorem 3.13. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB^* -triples not containing direct summands of rank smaller or equal than 3, or between two compact C^* -algebras. Suppose E has rank greater or equal than 5. Then there exists a surjective real linear isometry $T : E \rightarrow B$ satisfying $T|_{S(E)} = f$.*

Proof. Since the rank of E is greater or equal than 3, we can find three mutually orthogonal minimal tripotents e_1, e_2 and e_3 in E . For each $j \in \{1, 2, 3\}$, let $T_{e_j} : E_0^{**}(e_j) \cap E \rightarrow B_0^{**}(v_j) \cap B$ be the surjective real linear isometry given by Proposition 3.2(c), where $v_j = f(u_j)$. Proposition 3.1 and Corollary 3.8 assure that v_1, v_2 and v_3 are mutually orthogonal minimal tripotents in B and $f(e_1 + e_2 + e_3) = v_1 + v_2 + v_3$.

It is known that

$$(11) \quad E = \mathbb{C}e_1 \oplus (E_1(e_1) \cap E_1(e_2)) \oplus (E_1(e_1) \cap E_0(e_2)) \oplus E_0(e_1)$$

(just compare the joint Peirce decomposition given in [30, (1.12)]). Each x in E writes uniquely in the form $x = \lambda_x e_1 + x_{11} + x_{10} + x_0$, where $\lambda_x \in \mathbb{C}$, $x_{11} \in (E_1(e_1) \cap E_1(e_2))$, $x_{10} \in (E_1(e_1) \cap E_0(e_2))$, and $x_0 \in E_0(e_1)$. We define a mapping $T : E \rightarrow B$ given by

$$T(x) = T(\lambda_x e_1 + x_{11} + x_{10} + x_0) := T_{e_3}(\lambda_x e_1) + T_{e_3}(x_{11}) + T_{e_2}(x_{10}) + T_{e_1}(x_0),$$

where $x = \lambda_x e_1 + x_{11} + x_{10} + x_0$. The mapping T is well defined and real linear thanks to the uniqueness of the decomposition in (11) and the real linearity of T_{e_1} , T_{e_2} and T_{e_3} .

We claim that

$$(12) \quad T(e) = f(e), \text{ for every minimal tripotent } e \text{ in } E.$$

Let e be a minimal tripotent in E . Since E has rank bigger or equal than 5, we can find e_4 satisfying $e_4 \perp e_1, e_2, e_3, e$.

Since $e \in E_0(e_4)$, Corollary 3.10 implies that

$$(13) \quad f(e) = T_{e_4}(e).$$

Let us write $e = \lambda_e e_1 + e_{11} + e_{10} + e_0$, where $\lambda_e \in \mathbb{C}$, $e_{11} \in E_1(e_1) \cap E_1(e_2)$, $e_{10} \in (E_1(e_1) \cap E_0(e_2))$, and $e_0 \in E_0(e_1)$. Clearly, $\lambda_e e_1 \perp e_4$. By Peirce arithmetic $e_{11} \in E_1(e_1) \cap E_1(e_2) \subset E_2(e_1 + e_2) \perp e_4$, because $e_1 + e_2 \perp e_4$.

Now, having in mind that $e \perp e_4$, we deduce that

$$0 = L(e_4, e_4)(e) = L(e_4, e_4)(\lambda_e e_1 + e_{11} + e_{10} + e_0) = L(e_4, e_4)(e_{10}) + L(e_4, e_4)(e_0).$$

A new application of Peirce arithmetic, gives

$$L(e_4, e_4)(e_{10}) \in E_1(e_1), \text{ and } L(e_4, e_4)(e_0) \in E_0(e_1),$$

and thus

$$L(e_4, e_4)(e_0) = L(e_4, e_4)(x_{10}) = 0.$$

We have therefore shown that $\lambda_e e_1, e_{11}, e_{10}, e_0 \in E_0(e_4)$. Applying Lemma 3.11 to T_{e_4} and T_{e_3} (respectively, to T_{e_4} and T_{e_2} , and T_{e_4} and T_{e_3}) we obtain:

$$T(e) = T_{e_3}(\lambda_e e_1) + T_{e_3}(x_{11}) + T_{e_2}(x_{10}) + T_{e_1}(x_0)$$

$= T_{e_4}(\lambda_x e_1) + T_{e_4}(x_{11}) + T_{e_4}(x_{10}) + T_{e_4}(x_0) = T_{e_4}(e) =$ (by (13)) $= f(e)$,
which concludes the proof of the claim.

Let u be a finite rank tripotent in E , and let us write $u = u_1 + \dots + u_m$, where u_1, \dots, u_m are mutually orthogonal minimal tripotents in E . By Corollary 3.8 and (12) we have

$$f(u) = f(u_1) + \dots + f(u_m) = T(u_1) + \dots + T(u_m) = T(u).$$

This shows that $T(u) = f(u)$ for every finite rank tripotent u in E . Under these circumstances, Proposition 3.9 implies that $T(x) = f(x)$, for every $s \in S(E)$, which concludes the proof. \square

Let A be a C*-algebra. It is known that A is a weakly compact JB*-triple if and only if it is a compact JB*-triple if and only if A is a compact C*-algebra in the sense employed in [2, 49]. It is therefore known that $A = \oplus_j^{c_0} K(H_j)$, where each H_j is a complex Hilbert space. We can establish now a positive answer to Tingley's conjecture in the case of a surjective isometry between the unit spheres of two compact C*-algebras.

Theorem 3.14. *Let $f : S(A) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two compact C*-algebras. Then there exists a surjective real linear isometry $T : A \rightarrow B$ such that $T(x) = f(x)$, for every $x \in S(A)$. In particular, the same conclusion holds when A and B are the C*-algebras of compact operators on arbitrary complex Hilbert spaces.*

Proof. When A and B are finite dimensional, the conclusion follows from the main result in [47, Theorem 4.8] or from [46].

If A (or B) is the c_0 -sum of two or more $K(H_j)$, then the result follows from Theorem 3.12 (compare the comments in page 14). We can thus assume that $f : S(K(H_1)) \rightarrow S(K(H_2))$ is a surjective isometry. Since H_1 and H_2 are infinite dimensional the desired conclusion is a straight consequence of Theorem 3.13. \square

The results in this note cannot be applied in the study of surjective isometries between the unit spheres of two elementary JB*-triples of rank ≤ 4 . This question is left as an open problem, although some additional information is known. Let K be an elementary JB*-triple of rank 1, 2, 3 or 4. The second half of the Table 1 in [35, page 210] implies that K is one of the following Cartan factors:

- (a) A Cartan factor of type 1 of the form $L(H, K)$, with $\dim(K) \leq 4$;
- (b) A Cartan factor of type 2, with $\dim(H) \leq 9$;
- (c) A Cartan factor of type 3, with $\dim(H) \leq 4$;
- (d) A spin factor whose rank is always 2.

A Cartan factor C of rank 1 must be type 1 Cartan factor of the form $L(H, \mathbb{C})$, where H is a complex Hilbert space, or a type 2 Cartan factor with $\dim(H) = 1$ or 3 (it is known that C is JB*-triple isomorphic to a 1 or 3 dimensional complex Hilbert space). In any case, every rank-one Cartan factor is of the form $L(H, \mathbb{C})$. Let H_1 and H_2 be Hilbert spaces. G. Ding proved in [11] that every isometric surjective mapping $f : S(H_1) \rightarrow S(H_2)$ can be extended to a real linear isometric mapping $T : H_1 \rightarrow H_2$. Combining these arguments we get:

Corollary 3.15. *Let $f : S(H_1) \rightarrow S(H_2)$ be a surjective isometry between the unit spheres of two rank one JB*-triples. Then there exists a surjective real linear isometry $T : H_1 \rightarrow H_2$ such that $T(x) = f(x)$, for every $x \in S(H_1)$.*

In the case of finite dimensional JB^* -triples, the arguments above can be adapted to obtain a generalization of [46].

Proposition 3.16. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two finite dimensional JB^* -triples. Suppose that e_1 and e_2 are two orthogonal finite rank tripotents in E . Then $f(e_1)$ and $f(e_2)$ are two orthogonal finite rank tripotents in B with $f(e_1 + e_2) = f(e_1) + f(e_2)$. Furthermore, if e is a rank k tripotent in E then $f(e)$ is a rank k tripotent in B .*

Proof. Let us consider the tripotents $v = e_1 - e_2$ and $\tilde{v} = e_1 + e_2$. Since E and B are finite dimensional, by Tingley's theorem, we know that $f(e_1 - e_2) = f(u) = -f(-u) = -f(e_2 - e_1)$ (see [48]). By Proposition 3.2(d), $w_1 = f(e_1)$, $w_2 = f(e_2)$, $w = f(e_1 - e_2) = f(u)$ and $\tilde{w} = f(e_1 + e_2)$ are finite rank tripotents in B . Let T_{e_1} , T_{e_2} , T_v and $T_{\tilde{v}}$ denote the surjective real linear isometries given by Proposition 3.2(c).

Since e_1 , v and \tilde{v} lie in the face $\{e_1\}_{\prime\prime}$ and $e_1 = 1/2(v + \tilde{v})$, by Proposition 3.2(b) we have

$$\begin{aligned} w_1 = f(e_1) &= \frac{1}{2}(f(v) + f(\tilde{v})) = \frac{1}{2}(w + \tilde{w}) = \frac{1}{2}(f(v) + f(\tilde{v})) = \frac{1}{2}(f(e_1 - e_2) + f(e_1 + e_2)) \\ &= \frac{1}{2}(-f(e_2 - e_1) + w_2 + T_{e_2}(e_1)) = \frac{1}{2}(-w_2 - T_{e_2}(-e_1) + w_2 + T_{e_2}(e_1)), \end{aligned}$$

which shows that $T_{e_2}(e_1) = w_1 = f(e_1)$. Therefore

$$\tilde{w} = f(e_1 + e_2) = w_2 + T_{e_2}(e_1) = w_2 + w_1 = f(e_2) + f(e_1).$$

Replacing e_2 with $-e_2$ we prove

$$w = f(e_1 - e_2) = f(e_1) + f(-e_2) = f(e_1) - f(e_2) = w_1 - w_2.$$

we have therefore shown that $w_1 \pm w_2$, w_1 and w_2 are tripotents in B , and hence $w_1 \perp w_2$ (compare (4)).

Since by Proposition 3.1(c) f maps minimal tripotents to minimal tripotents, we deduce from the first statement that f maps tripotents of rank k to tripotents of rank k . \square

When in the proof of Corollary 3.10 and Lemma 3.11, we replace Corollary 3.8 with Proposition 3.16 we obtain:

Corollary 3.17. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two finite dimensional JB^* -triples. Let e be a finite rank tripotent in E , $u = f(e)$, and let $T_e : E_0(e) \rightarrow B_0(u)$ be the surjective real linear isometry given by Proposition 3.2(c). Then $f(x) = T_e(x)$ for all $x \in S(E_0(e))$. \square*

Lemma 3.18. *$f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two finite dimensional JB^* -triples. Let e_1 and e_2 be two orthogonal finite rank tripotents in E , and let T_{e_1} and T_{e_2} be the maps given by Proposition 3.2(c). Then*

$$T_{e_1}(x) = T_{e_2}(x) \text{ for all } x \in E_0(e_1) \cap E_0(e_2).$$

\square

An obvious adaptation of the arguments in the proof of Theorem 3.19 can be now applied to establish a generalization of the main result in [46].

Theorem 3.19. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry between the unit spheres of two finite dimensional JB^* -triples. Suppose E has rank greater or equal than 5. Then there exists a surjective real linear isometry $T : E \rightarrow B$ satisfying $T|_{S(E)} = f$. \square*

We do not know the answer in the remaining cases.

Problem 3.20. *Let C and B be Cartan factors of rank ≤ 4 . Suppose $f : S(C) \rightarrow S(B)$ is a surjective isometry. Does f extend to a surjective real linear isometry from C into B ?*

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